Workshop 1: More about events

Wednesday, January 10th, 2021

1 Inclusion-exclusion principle

Exercise 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove by induction in $n \geq 2$ that for any events $A_1, ..., A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_1 \cap A_j \cap A_k) + \cdots + (-1)^n P(A_1 \cap \cdots \cap A_n).$$

Bonus: Find interesting applications of this principle in connection with your interests.

2 $\liminf$ and $\limsup$

Let $(x_n)_n$ be a sequence of real numbers. One defines

$$\limsup_n x_n := \inf_m \left(\sup_{n \geq m} x_n\right) = \lim_m \downarrow \left(\sup_{n \geq m} x_n\right), \tag{1}$$

$$\liminf_n x_n := \sup_m \left(\inf_{n \geq m} x_n\right) = \lim_m \uparrow \left(\inf_{n \geq m} x_n\right), \tag{2}$$

Remark that if $z < \limsup x_m$ then $z < x_n$ 'infinitely often".

More generally, for a sequence of events $(E_n)_n$, we introduce the events

$$\begin{aligned}(E_n, \text{ i.o.}) &:= (E_n \text{ happens infinitely often}) \\ &:= \limsup_n E_n \\ &:= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} E_n\end{aligned}$$

and

$$\begin{aligned}(E_n, \text{ ev.}) &:= (E_n \text{ happens infinitely often}) \\ &:= \liminf_n E_n \\ &:= \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} E_n\end{aligned}$$

Exercise 2. Prove that

$$\left(\liminf_n E_n\right)^c = \limsup_n E_n^c.$$

Exercise 3 (Cf. Williams, 2.9)**.** For an event E, define the indicator function I_E on Ω via

$$I_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}.$$

Let $(E_n)_n$ be a sequence of events. Prove that, for each ω,

$$I_{\limsup_n E_n}(\omega) = \limsup_n I_{E_n}(\omega),$$

and establish the corresponding result for lim infs.

Exercise 4. Prove that, for any sequence of events $(E_n)_{n\in\mathbb{N}}$,

$$P\left(\limsup_n E_n\right) \geq \limsup_n P(E_n),$$

and

$$P\left(\liminf_n E_n\right) \leq \liminf_n P(E_n).$$

Exercise 5 (First Borel Cantelli Lemma)**.** Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of events such that $\sum_n P(E_n) < \infty$. Then

$$P(\limsup_n E_n) = 0.$$

If you want to dive deeper into measure theory:

Exercise 6. An **outer measure** μ_* on Ω is a function $\mu_* : 2^\Omega \to [0,\infty]$ such that

(a) $\mu_*(\emptyset) = 0$;

(b) if $A \subset B \subset \Omega$, then $\mu_*(A) \leq \mu_*(B)$;

(c) if $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of Ω,

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu_*(A_n).$$

A set A is called **μ_*-measurable** if for all $B \subset \Omega$,

$$\mu_*(B) = \mu_*(B \cap A) + \mu_*(B \cap A^c).$$

1. Prove that the μ_*-measurable sets form a σ-algebra.
2. Prove that μ_* restricted to the σ-algebra of μ_*-measurable sets is a measure.
3. Let $F : \mathbb{R} \to \mathbb{R}$ be bounded, non-decreasing, and right continuous, such that $\lim_{x\to-\infty} F(x) = 0$. Prove that the map $\nu_* : 2^\mathbb{R} \to [0,\infty]$ defined by

$$\nu_*(A) = \inf\left\{\sum_n (F(b_n) - F(a_n)) \ : \ \{(a_n, b_n]\}_n \text{ intervals such that } A \subset \bigcup_n (a_n, b_n]\right\}$$

is an outer measure on $\mathbb{R}$.

4. The equality $\nu_*((-\infty, x]) = F(x)$ holds (you do not need to prove this), hence

$$\nu_*((a_n, b_n]) = \nu_*((-\infty, b_n]) - \nu_*((-\infty, a_n]) = F(b_n) - F(a_n).$$

Prove that the Borel subsets of $\mathbb{R}$ are ν_*-measurable. (Hint: Prove first that the intervals of the form $(-\infty, c]$ are ν_*-measurable.)