Workshop 2

Wednesday, January 17th, 2021

1 Independence

Definition 1 (Independence)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space.

- Sub-σ-algebras $\mathcal{G}_1$, $\mathcal{G}_2, \cdots$ of $\mathcal{F}$ are called independent if for every finite set $I = \{i_1, ..., i_n\}$ of distinct indexes and every $G_i \in \mathcal{G}_i$, $i \in I$,

$$P(\bigcap_{k=1}^{n} G_{i_k}) = \prod_{k=1}^{n} P(G_{i_k}).$$

- Events E_1, $E_2, \cdots$ in $\mathcal{F}$ are called independent if the sub-σ-algebras $\sigma(E_1)$, $\sigma(E_2)$, ... are independent.

- Random variables X_1, X_2, ... are independent if the sub-σ-algebras $\sigma(X_1)$, $\sigma(X_2)$, ... are independent.

Exercise 1 (A way of verifying independence, cf. Williams 4.2)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space.

1. Suppose $\mathcal{G}$ and $\mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$ and that $\mathcal{I}$ and $\mathcal{J}$ are π-systems such that $\mathcal{G} = \sigma(\mathcal{I})$ and $\mathcal{H} = \sigma(\mathcal{J})$. Prove that $\mathcal{G}$ and $\mathcal{H}$ are independent if and only if

$$\forall I \in \mathcal{I}, \forall J \in \mathcal{J}, \quad P(I \cap J) = P(I)P(J). \tag{1}$$

   (Hint: First fix J and think about the functions $P_1(I) = P(I \cap J)$ and $P_2(I) = P(I)P(J)$ on $\mathcal{I}$.)

2. Deduce that (real-valued) random variables X and Y are independent if and only if

$$\forall (x, y) \in \mathbb{R}^2, \quad P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y). \tag{2}$$

2 Second Borel-Cantelli lemma

Exercise 2 (The lemma)**.** Suppose that the events $(E_n)_{n\in\mathbb{N}}$ are independent. Prove that if $\sum_n P(E_n) = \infty$ then $P(\limsup_n E_n) = 1$.
(Hint: You'll have to use that $\prod_{m\leq n\leq r}(1 - p_n) \leq \exp(-\sum_{n=m}^{r} p_n)$ whenever $0 \leq p_n \leq 1$.)

Exercise 3 (An application, cf. Williams E4.4)**.** Suppose that a coin with probability p of heads is tossed repeatedly. Let A_k be the event that a sequence of k (or more) consecutive heads occurs amongst tosses numbered 2^k, $2^k + 1$, $2^k + 2, ..., 2^{k+1} - 1$. Prove that

$$P(A_k, \text{ i.o.}) = \begin{cases} 1 & \text{if } p \geq 1/2 \\ 0 & \text{if } p < 1/2 \end{cases}. \tag{3}$$

3 Tail σ-algebra

Let X_1, X_2, X_3, ... be a sequence of random variables. Define

$$\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, ...), \quad \mathcal{T} = \bigcap_{n=1}^{\infty} \mathcal{T}_n. \tag{4}$$

The resulting collection of sets $\mathcal{T}$ is called the **tail σ-algebra** of the sequence $(X_n)_{n\geq 1}$.

Exercise 4. Prove that $\mathcal{T}$ contains the event "$\sum_{n\geq 1} X_n$ converges".

Exercise 5 (Kolmogorov's 0-1 law, cf. Williams 4.11)**.** Suppose that the variables $(X_n)_{n\geq 1}$ are independent.

1. Prove that $\mathcal{T}$ is independent from $\mathcal{T}$.
   (Hint: Prove first that $\mathcal{F}_n = \sigma(X_1, ..., X_n)$ and $\mathcal{T}_n = \sigma(X_{n+1}, ...)$ are independent. Take $n \to \infty$.)

2. Deduce that for any $F \in \mathcal{T}$, $P(F)$ equals 0 or 1.

3. Prove that if $h : \Omega \to \mathbb{R}$ is measurable, then there exists $c \in [-\infty, \infty]$ such that $P(h = c) = 1$.

4 Some suggestions for independent work

- Exercise E4.1 in Williams' book is an extension of Exercise 1 here to three σ-algebras.
- If you're interested in number theory, consider solving exercise E4.2 in William's book.
- Exercises E4.4-E4.8 in that same book are good practice for the Borel-Cantelli lemmas.