Workshop 5

August 3, 2024

1 Weak convergence

Let Prob($\mathbb{R}$) denote the set of probability distributions on the real line $\mathbb{R}$, and $C_b(\mathbb{R})$ denote the set of *bounded and continuous* functions with domain and codomain $\mathbb{R}$.

We say that a sequence $(\mu_n)_n \subset \mathrm{Prob}(\mathbb{R})$ converges weakly to $\mu \in \mathrm{Prob}(\mathbb{R})$ if for all $h \in C_b(\mathbb{R})$ one has $\mu_n(h) \to \mu(h)$ as $n \to \infty$. This is denoted $\mu_n \xrightarrow{w} \mu$.

Exercise 1. Let $(X_n)_{n\in\mathbb{N}}$ and X be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

1. $X_n \to X$ almost surely implies that $F_{X_n} \xrightarrow{w} F_X$.
2. $X_n \to X$ in probability implies that $F_{X_n} \xrightarrow{w} F_X$.

Exercise 2 (Cf. Williams E18.3)**.** Let X and Y be random variables taking values in $[0, 1]$. Suppose that

$$\mathbb{E}(X^k) = \mathbb{E}(Y^k), \quad \text{for } k = 0, 1, 2, ...$$

Prove that

1. $\mathbb{E}p(X) = \mathbb{E}p(Y)$ for every polynomial p,
2. $\mathbb{E}f(X) = \mathbb{E}f(Y)$ for every continuous function f on $[0, 1]$,
3. $P(X \le t) = P(Y \le t)$ for every t in $[0, 1]$.

2 Characteristic functions

Exercise 3 ("The law of small numbers")**.** For $\lambda > 0$, suppose given a sequence $(F_n)_{n\in\mathbb{N}}$ such that, for every $n > \lambda$, F_n is the cumulative distribution function (cdf) of $\mathrm{Bin}(n, \lambda/n)$. Prove, using characteristic functions, that F_n converges weakly to the cdf F of a Poisson random variable with parameter λ.[1]

Exercise 4. Prove that if Z has a uniform distribution on the interval $[-1, 1]$ i.e. $Z \sim U[-1, 1]$, then $\phi_Z(\theta) = (\sin\theta)/\theta$. Moreover, prove that there do not exist i.i.d. random variables X and Y such that $X - Y \sim U[-1, 1]$.

More problems: Exercises E18.3-E18.6 in Williams guide you through a solution to the *Hausdorff moment problem* (existence of a $[0, 1]$-valued random variable with prescribed moments). Exercises EA13.1 and EA13.2 are about modes of convergence.

[1] In other words, if you toss n coins with an asymptotically small probability of heads λ/n (so that the expectation remains constant, λ), then the number of heads behaves asymptotically like a Poisson distribution with parameter λ. Often this explains the appearance of the Poisson distribution in applications.