Workshop 6: Conditional expectation

August 3, 2024

1 Some foundational (and relatively simple) exercises

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Prove that if $X \geq 0$, then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.

Exercise 2 (Conditional version of Fatou's lemma)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Prove that

$$\mathbb{E}(\liminf_n X_n|\mathcal{G}) \leq \liminf_n \mathbb{E}(X_n|\mathcal{G}) \quad \text{a.s.} \tag{1}$$

(Hint: You can simply generalize the proof of 5.4 in Williams' boook. Justify every step using the properties of conditional expectations that we discussed in class, cf Williams 9.7.)

2 Conditionaning via disintegration

The results stated in Exercise (3) are used to solve Exercises (4) and (5).

Exercise 3. Let $(E_X, \mathcal{B}_X, \mu)$ and $(E_U, \mathcal{B}_U, \nu)$ be σ-finite measure spaces. For instance, the real line equipped with the Lebesgue measure λ on Borel sets, a discrete set equipped with the counting measure # on all subsets, or a combination of both.

We consider them to be co-domains of random variables $X : \Omega \to E_X$ and $U : \Omega \to E_Y$ on a probability triple $(\Omega, \mathcal{F}, P)$.

1. Suppose a probability measure $\rho \ll (\mu \times \nu)$ is given, which we interpret as the joint law of (X, U); let $f_{X,U}(x, u)$ be its density w.r.t. $\mu \times \nu$. Use Fubini's theorem to show that $x \mapsto f_X(x) := \int_{E_U} f_{X,U}(x, u)\,\mathrm{d}\nu(u)$ and $u \mapsto f_U(u) = \int_{E_X} f_{X,U}(x, u)\,\mathrm{d}\mu(x)$ are measurable, and that $f_X.\mu$ (resp. $f_U.\nu$) is a probability measure on E_X (resp. E_U).

2. With the assumptions and notations of the previous point, define the conditional density function

$$f_{U|X}(u|x) = \begin{cases} \frac{f_{X,U}(x,u)}{f_X(x)} & \text{if } f_X(x) > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Verify that

$$P_{\sigma(\pi_1)} \equiv P_X : (E_X \times E_U) \times \sigma(\pi_2) \to [0, 1], \quad ((x, u), E_X \times B) \mapsto \int_B f_{U|X}(u', x)\,\mathrm{d}\nu(u') \tag{3}$$

is a regular conditional probability on $\sigma(\pi_2)$, the algebra generated by the projection on the second component, given $\sigma(\pi_1)$.

Convince yourself that this is equivalent to a regular conditional probability

$$P_{\sigma(X)} : \Omega \times \sigma(U) \to [0, 1]. \tag{4}$$

3. Conversely, we can define a joint law for (X, U) starting with a probability measure $f_U . \mathrm{d}\nu$ and a (probabilistic) kernel $K : E_U \times \mathcal{B}_X \to [0,1]$. By definition, the map K is such that for every $B \in \mathcal{B}_X$, $u \mapsto K(u, B)$ is measurable and for every $u \in E_U$, $B \mapsto K(u, B)$ is a probability measure.

   If we have conditional probability density functions $u \mapsto f_{X|U}(\cdot, u)$, the correspondence $K(u, B) = \int_B f_{X|U}(u, x)\,\mathrm{d}\mu(x)$ defines (in most practical cases) such a kernel.

   The induced measure on the product space $E_X \times E_U$ given by the formula

$$\forall S \in \mathcal{B}_X \otimes \mathcal{B}_U, \quad \tau(S) = \int_{\pi_2(S)} K(u, S_u) f_U(u)\,\mathrm{d}\nu(u), \tag{5}$$

   where S_u denotes the *slice* $\{x \in E_X \mid (x, u) \in S\}$.

   Why is τ a probability measure?

Exercise 4. Suppose $E_U = [0, 1]$, $\nu = \lambda$, $U \sim \text{Uniform}(0, 1)$, $E_X = \{0, 1, ..., n\}$, $\mu = \#$, and $X \sim \text{Bin}(n, U)$ in the sense that

$$f_{X|U}(x, u) = \binom{n}{k} u^k (1-u)^{n-k} \quad \text{for } x \in E_X \tag{6}$$

1. Find the density $f_{X,U}$ w.r.t. $\# \times \lambda$ and the conditional density $f_{U|X}$ w.r.t. λ.
2. Prove that

$$\mathbb{E}(s^X) = \frac{1}{n+1} \sum_{k=0}^{n} s^k. \tag{7}$$

   and deduce that when $0 \le k \le n$,

$$\int_0^1 u^k (1-u)^{n-k}\,\mathrm{d}u = (n+1)^{-1} \binom{n}{k}^{-1}. \tag{8}$$

Exercise 5. Let X have Poisson distribution with (random) parameter Λ, where Λ is exponential with parameter μ.[1]

1. Compute the probability generating function of X, i.e. $G_X(s) = \mathbb{E}(s^X)$.
2. Show that X follows a geometric distribution.[2] What is the value of the parameter?
3. Find $f_{\Lambda|X}$, the density of the corresponding regular conditional probability with respect to the Lebesgue measure.

3 A more advanced exercise

Exercise 6 (Bonus - 1 point)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability triple and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Let X and Y be real valued random variables such that X is $\mathcal{G}$ measurable and $\sigma(Y)$ is independent of $\mathcal{G}$; we denote by $\Lambda_{X,Y}$ their joint law Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function in $\mathcal{L}^1(\mathbb{R}^2, B(\mathbb{R}^2), \Lambda_{X,Y})$. Define $\gamma^h(x) = \mathbb{E}h(x, Y)$. Prove that $\gamma^h(X)$ is a version of the conditional expectation $\mathbb{E}(h(X, Y)|\mathcal{G})$. (Hint: Use the Monotone Class Theorem; compare with Williams' proof of Fubini's theorem. Caveat: this result is not implied those in Section 9.10 of Williams' book.)

[1] A real-valued random variable Z has/follows an exponential distribution with parameter λ if its p.d.f. with respect to the Lebesgue measure is $\lambda e^{-\lambda x} I_{\{x \ge 0\}}$.

[2] An $\mathbb{N}_0$-valued random variable Y follows a geometric distribution with parameter p if $P(Y = k) = (1-p)^k p$. It corresponds to the number of "failures" before the first "success" in a repeated sequence of i.i.d. Bernoulli trials with parameter p.