Some exercises that involve martingales

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Exercise 1 (Azuma's inequality)**.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{n\in\mathbb{N}_0}, P)$ be a filtered space and let $(Y_n)_{n\in\mathbb{N}_0}$ be a martingale with respect to that filtration. We suppose that there is a positive constant K such that $P(|Y_n - Y_{n-1}| \leq K) = 1$, for all $n \in \mathbb{N}$. In this exercise, you'll prove that

$$\forall n \in \mathbb{N}, \quad P(Y_n - Y_0 \geq x) \leq \exp\left(-\frac{1}{2}\frac{x^2}{nK^2}\right), \tag{1}$$

which is a martingale analogue of Hoeffding's inequality.

1. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Pick $\psi > 0$. If D is a random variable such that $\mathbb{E}(D|\mathcal{G}) = 0$ and $P(|D| \leq 1) = 1$, then
$$\mathbb{E}(e^{\psi D}|\mathcal{G}) \leq e^{\frac{1}{2}\psi^2}. \tag{2}$$
(Hint: Since $e^{\psi x}$ is convex in x,
$$e^{\psi x} \leq \frac{1-x}{2}e^{-\psi} + \frac{x+1}{2}e^{\psi} \quad \text{whenever } -1 \leq x \leq 1. \tag{3}$$
Take conditional expectations and use your best knowledge of elementary calculus to bound the resulting expression.)

Remember that Markov's inequality implies that, for all $\theta > 0$,

$$P(Y_n - Y_0 \geq x) \leq e^{-\theta x}\mathbb{E}(e^{\theta(Y_n - Y_0)\cdot}) \tag{4}$$

2. Show that, for any $\theta > 0$,
$$\mathbb{E}(e^{\theta(Y_n - Y_0)}|\mathcal{F}_{n-1}) \leq e^{\theta(Y_{n-1}-Y_0)} \exp\left(\frac{1}{2}\theta^2 K^2\right). \tag{5}$$

3. Conclude from (5) that
$$\mathbb{E}(e^{\theta(Y_n - Y_0)}) \leq \exp\left(\frac{1}{2}n\theta^2 K^2\right). \tag{6}$$

Combined with (4), this last result implies that

$$P(Y_n - Y_0 \geq x) \leq \exp\left(-\theta x + \frac{1}{2}n\theta^2 K^2\right) \tag{7}$$

4. Prove (1) finding an optimal θ. Then show that
$$P(|Y_n - Y_0| \geq x) \leq 2\exp\left(-\frac{1}{2}\frac{x^2}{nK^2}\right). \tag{8}$$

Exercise 2 (Bellman's Optimality Principle, Williams, E10.2)**.** Your winnings per unit stake on game n are ϵ_n, where $(\epsilon_n)_{n\in\mathbb{N}}$ are i.i.d. random variables such that $P(\epsilon_i = 1) = p > 1/2$ and $P(\epsilon_i = -1) = q := 1-p$, for all i. Your stake C_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n-1$. Then, for all $n \geq 1$,

$$Z_{n+1} = Z_n - C_{n+1} + (1+\epsilon_{n+1})C_{n+1}. \tag{9}$$

We suppose that Z_0, the fortune at time 0, is a given constant. Your object is to maximize the expected *interest rate* $\mathbb{E}\log(Z_N/Z_0)$, where N is a given integer representing the length of the game.

Set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(\epsilon_1, ..., \epsilon_n)$.

Show that if C is any "reasonable" previsible strategy (w.r.t. $\{\mathcal{F}_n\}_n$), then $\log Z_n - n\alpha$ is a supermartingale, where α denotes the 'relative entropy'

$$\alpha = p\log p + q\log q + \log 2, \tag{10}$$

and therefore $\mathbb{E}\log(Z_N/Z_0) \leq N\alpha$. Find then a previsible strategy C for which $\log Z_n - n\alpha$ *is a martingale* and show that this strategy maximizes $\mathbb{E}\log(Z_N/Z_0)$.

(Hints: Here "reasonable" means that $\log Z_n$ remains integrable for all n i.e. Z_n remains far enough from 0.

Write $C_{n+1} = K_{n+1}Z_n$, for certain previsible process $(K_n)_n$ with values in $[0,1]$; prove first the elementary inequality

$$\forall x \in [0,1], \quad p\log(1+x) + q\log(1-x) \leq \alpha. \tag{11}$$

For which value(s) of x the equality holds?

You may use a result stated in a previous workshop: *Let $(\Omega, \mathcal{F}, P)$ be a probability triple and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Let X and Y be real valued random variables such that X is $\mathcal{G}$ measurable and $\sigma(Y)$ is independent of $\mathcal{G}$; we denote by $\Lambda_{X,Y}$ their joint law. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function in $\mathcal{L}^1(\mathbb{R}^2, B(\mathbb{R}^2), \Lambda_{X,Y})$. Define $\gamma^h(x) = \mathbb{E}h(x,Y)$. The function $\gamma^h(X)$ is a version of the conditional expectation $\mathbb{E}(h(X,Y)|\mathcal{G})$.*)

Exercise 3 (Williams, E10.5)**.** Suppose that T is a stopping time such that for some $N \in \mathbb{N} = \{1, 2, 3, ...\}$ and some $\epsilon > 0$, we have, for every $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$:

$$P(T \leq n + N|\mathcal{F}_n) > \epsilon, \quad \text{a.s.} \tag{12}$$

Prove by induction that for every $k \in \mathbb{N}$,

$$P(T > kN) \leq (1-\epsilon)^k \tag{13}$$

and deduce from this that $\mathbb{E}(T) < \infty$.

(Hint: Use that $P(T > kN) = P(T > kN; T > (k-1)N) = \mathbb{E}(I_{\{T>kN\}}I_{\{T>(k-1)N\}})$.)

Exercise 4 (Williams, E10.7)**.** Suppose that $(X_n)_{n\geq 0}$ are i.i.d. random variables with

$$P(X_i = 1) = p, \quad P(X_i = -1) = 1-p =: q, \quad 0 < p < 1, \tag{14}$$

and $p \neq q$. Suppose that a and b are integers such that $0 < a < b$. Define $S_0 = a$ and $S_n = a + X_1 + \cdots + X_n$ for $n \geq 1$.[1] We introduce the filtration given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, ..., X_n)$ whenever $n \geq 1$, as well as the stopping time $T := \inf\{n \in \mathbb{N} \mid S_n = 0 \text{ or } S_n = b\}$.

[1] This is different from the definition introduced in class: in principle S_n could go outside $[0, b]$, but if S_n is a martingale then the stopped process $S_{T\wedge n}$ too.

1. Find N and ϵ such that T satisfies the condition (12) in Exercise 3.
2. Prove that $N_n = S_n - n(p-q)$ is a martingale.
3. We already showed that $M_n = (q/p)^{S_n}$ is also a martingale.
   Deduce rigorously, using the preceding results, the value of $P(S_T = 0)$ and $\mathbb{E}(T)$