Workshop 7: Martingale convergence

August 3, 2024

1 Galton-Watson process

Exercise 1. A branching process $Z = \{Z_n : n \geq 0\}$ is constructed in the usual way. Thus, a family $\{X_k^{(n)} : n, k \in \mathbb{N}\}$ of i.i.d. $\mathbb{N}_0$-valued random variables is supposed given. we define $Z_0 := 0$ and then define recursively, for each $n \geq 0$,

$$Z_{n+1} := X_1^{(n+1)} + \cdots X_{Z_n}^{(n+1)}. \tag{1}$$

Assume that if X denotes any one of the $X_k^{(n)}$, then

$$\mu := \mathbb{E}(X) < \infty \quad \text{and} \quad 0 < \sigma^2 := \text{Var}(X) < \infty. \tag{2}$$

1. Prove that $M_n := Z_n/\mu^n$ defines a martingale M relative to the filtration $\mathcal{F}_n = \sigma(Z_1, ..., Z_n)$.
2. Show that M_∞ exists and is finite, almost surely.
3. Let $f(\theta) = \mathbb{E}(\theta^X)$ and $f_n(\theta) = \mathbb{E}(\theta^{Z_n})$. Prove that $f_{n+1} = f_n \circ f$ and hence that f_n is the n-fold composition of f. Moreover, prove that $L(\lambda) = \mathbb{E}(\exp(-\lambda M_\infty)$ is continuous on $(0, \infty)$ and satisfies the functional equation
$$L(\lambda\mu) = f(L(\lambda)). \tag{3}$$
Hint: For continuity, think about sequential continuity. To establish (3), prove first that $\mathbb{E}(\exp(-\lambda M_n) = f_n(\exp(-\lambda/\mu^n))$, where $f_n(\theta) = \mathbb{E}(\theta^{Z_n})$.
4. Show that
$$\mathbb{E}(Z_{n+1}^2 | \mathcal{F}_n) = \mu^2 Z_n^2 + \sigma^2 Z_n \tag{4}$$
and deduce that M is bounded in $\mathcal{L}^2$ (this is, that $\sup_n \|M_n\|_2 < \infty$) if and only if $\mu > 1$. Use this to prove that, when $\mu > 1$, $\lim_n \mathbb{E}(M_n) = \mathbb{E}(M_\infty)$.
5. Prove that
$$\text{Var}(M_\infty) = \sigma^2(\mu(\mu-1))^{-1}. \tag{5}$$
Hint: See section 14.11 in Williams' book.

2 Uniform integrability

Exercise 2. Prove that a class $\mathcal{C}$ of random variables (on a probability triple $(\Omega, \mathcal{F}, P)$...) is uniformly integrable if and only if the following conditions hold (simultaneously):

1. $\mathcal{C}$ is bounded in $\mathcal{L}^1$, i.e. $A := \sup\{\mathbb{E}(|X|) : X \in \mathcal{C}\} < \infty$;
2. for every $\epsilon > 0$, there exists $\delta > 0$ such that if $F \in \mathcal{F}$ satisfies $P(F) < \delta$ and $X \in \mathcal{C}$, then $\mathbb{E}(|X|; F) := \int_F |X| \, \mathrm{d}P < \epsilon$.

Hint for 'if': For $X \in \mathcal{C}$, $P(|X| > K) \leq K^{-1}A$.
Hint for 'only if': $\mathbb{E}(|X|; F) \leq \mathbb{E}(|X|; |X| > K) + KP(F)$.