Problem set 1:
Quotient topology; Homotopies

Due on Thursday, October 6th, at 11:59pm.

Exercise 1. There are four common definitions of the torus $\mathbb{T}^2$:

1. as $\mathbb{R}^2/\mathbb{Z}^2$, the plane module the equivalence relation given by $(x, y) \sim (u, w)$ if and only if $x - u$ and $y - w$ are both integers;
2. as a square with opposite edges identified as in Figure 1;
3. as the product $\mathbb{S}^1 \times \mathbb{S}^1$, where $\mathbb{S}^1$ is the set of solutions of $x^2 + y^2 = 1$ in $\mathbb{R}^2$ (equivalently, the set of $z \in \mathbb{C}$ such that $|z| = 1$);
4. the set of solutions of $(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$ in $\mathbb{R}^3$, for given $R, r > 0$.

Show that these are all homeomorphic to one another (try to be as explicit as possible).

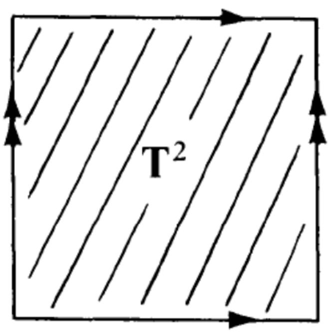

Figure 1: A square with opposite sides identified.

Exercise 2. Show that the projective plane is homeomorphic to the mapping cone of the map $z \mapsto z^2$ of the unit circle in the complex numbers to itself.

Exercise 3. Let $X = \mathbb{S}^1 \vee \mathbb{S}^1$ be the "one-point union" of two circles (see Bredon, Problem 8 in Chapter I, Section 13). Let Y be the union of the unit circle (in $\mathbb{R}^2$) with the segment $\{(0, y) : -1 < y < 1\}$. Show that X and Y are homotopically equivalent.

Exercise 4. A subset A of a topological space X is called a *retract* if there exists a continuous map $r : X \to X$ such that $r(X) = A$ and $r|_A = \mathrm{id}_A$.
Show that a retract of a contractible space is contractible.