Problem set 3

Due on Thursday, October 20th, at 11:59pm.

The first exercise is about chain complexes.
The second and third exercises are about relative homology, introduced by Hatcher in p. 115.

Exercise 1 (Homological algebra)**.** A *chain complex of abelian groups* C is, by definition, a sequence of abelian groups $(C_n)_{n\in\mathbb{N}}$ and boundary map $(\partial_n : C_n \to C_{n-1})_{n\in\mathbb{N}}$, such that $\partial_n \circ \partial_{n+1} = 0$ for each $n \in \mathbb{N}$ (sometimes written $\partial^2 = 0$). The homology $H(C)$ of the complex C is the family of abelian groups $H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.
A *chain map* (or *chain transformation*) $f : C \to C'$ is a family of homomorphisms $f_n : C_n \to C'_n$, one for each $n \in \mathbb{N}$, such that $\partial'_n f_n = f_{n-1}\partial_n$ for all n. In particular, the chain map $1_C : C \to C$ is the identity homomorphism for each n.
A *chain homotopy* s between two chain maps $f, g : K \to K'$ is a family of homomorphisms $s_n : K_n \to K'_{n+1}$, one for each $n \in \mathbb{N}$, such that

$$\partial'_{n+1} s_n + s_{n-1}\partial_n = f_n - g_n. \qquad (1)$$

We write $s : f \simeq g$.
Finally, a chain map $f : C \to C'$ is called a *chain equivalence* if there is another chain map $h : C' \to C$ and homotopies $s : hf \simeq 1_C$ and $t : fh \simeq 1_{C'}$. We say that C and C' are chain equivalent if such an f exists.

1. Show that chain complexes and chain maps form a category. What are the objects? Arrows? Compositions? Identities? Show that compositions are associative and that identities are neutral elements for composition.
2. Prove that chain homotopies $s : f \simeq g : C \to C'$ and $s' : f' \simeq g' : C' \to C''$ yield a composite chain homotopy between $f'f$ and $g'g$.
3. Prove that *chain equivalence* is indeed an equivalence relation among chain complexes.

Exercise 2. Show that if A is a retract of X then the map $\iota_\# : H_n(A) \to H_n(X)$ induced by the inclusion $\iota : A \to X$ is injective, for every $n \in \mathbb{N}$.

Exercise 3. For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.