Problem set 7: Differential forms and De Rham cohomology

Due on Thursday, November 17th, at 11:59pm.

Exercise 1 (Exterior derivative)**.** Let U be an open subset of $\mathbb{R}^n$. Prove that there is a unique operator $d : \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying

1. $d(\omega + \eta) = d\omega + d\eta$,
2. if $\omega \in \Omega^p(U)$ and $\eta \in \Omega^q(U)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$,
3. for $f \in \Omega^0(U)$ and X a vector field, $df(X) = X(f)$, and
4. for $f \in \Omega^0(U)$, $d(df) = 0$.

Show that d is the operator introduced in class (in terms of the basis $(dx_1, ..., dx_n)$ dual to the canonical basis).

Exercise 2 (Differential forms on $\mathbb{R}^2$)**.** Consider the 1-form

$$\omega(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}\right) dx_1 + \left(\frac{x_1}{x_1^2 + x_2^2}\right) dx_2. \tag{1}$$

defined on $\mathbb{R}^2 \setminus \{0\}$. Show that $d\omega = 0$ but that there is no smooth function (0-form) g such that $dg = \omega$. Conclude that $H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$.
(Hint: By contradiction.)

Exercise 3 (Mayer-Vietoris)**.** Let U_1 and U_2 be open set of $\mathbb{R}^n$ with union $U = U_1 \cup U_2$. For $\nu = 1, 2$, let $i_\nu : U_\nu \to U$ and $j_\nu : U_1 \cap U_2 \to U_\nu$ be the corresponding inclusions. Prove that the sequence

$$0 \longrightarrow \Omega^p(U) \xrightarrow{I^p} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{J^p} \Omega^p(U_1 \cap U_2) \longrightarrow 0 \tag{2}$$

is exact, where $I^p(\omega) = (i_1^*(\omega), i_2^*(\omega))$ and $J^p(\omega_1, \omega_2) = j_1^*(\omega_1) - j_2^*(\omega_2)$.[1]
(Hint: To prove surjectivity of I^p, consider a partition of unity subordinated to $\{U_1, U_2\}$.)

Exercise 4 (Integration)**.** Prove that if $\theta : M \to N$ is a diffeomorphism of oriented n-manifolds, which preserves orientation, then

$$\int_M \theta^* \omega = \int_N \omega \tag{3}$$

for any n-form ω on N with compact support.
(Hint: See Sections 2 and 3 in Chapter V of Bredon's book.)

[1] By homological algebra, this implies that there is a long exact sequence in cohomology

$$\cdots \to H^p(U) \to H^p(U_1) \oplus H^p(U_2) \to H^p(U_1 \cap U_2) \to H^{p+1}(U) \to \cdots$$