Problem set 8: Universal Coefficient Theorem

Due on **Sunday**, November 26th, at 11:59pm.

4 points per exercise

Exercise 1 (Torsion)**.**

1. Show that a free group is projective.
2. Show that any abelian group G has a projective resolution of the form
$$0 \to R \to F \to G \to 0.$$
Define explicitly the involved groups and maps.

Given the projective resolution $0 \to R \to F \to A \to 0$ of an abelian group A, we define $\text{Tor}(A, B)$ via the exactness of
$$0 \to \text{Tor}(A, B) \to R \otimes B \to F \otimes B \to A \otimes B \tag{1}$$

3. Show that $\text{Tor}(A, B) \cong \text{Tor}(B, A)$.

Exercise 2. Show that Ext and Tor, defined respectively via injective and projective resolutions, commute with finite direct sums. This is, that $\text{Ext}(A \oplus B, G) \cong \text{Ext}(A, G) \oplus \text{Ext}(B, G)$ and $\text{Ext}(A, G \oplus H) \cong \text{Ext}(A, G) \oplus \text{Ext}(A, H)$, and similarly for Tor.

The next two exercises are taken from Section V.7 in Bredon's book.

Exercise 3. Given that the Klein bottle K_2 has $H_0(K_2) \cong \mathbb{Z}$, $H_1(K_2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and all other integral homology groups zero, compute the homology and cohomology of K_2 in all dimensions for coefficients in $\mathbb{Z}$, and in $\mathbb{Z}_p$ for all primes p.

Exercise 4. Suppose $f, g : X \to Y$ are maps such that $f_* = g_* : H_*(X; Z) \to H_*(Y; Z)$. There are cases in the literature of the Universal Coefficient Theorem being cited as implying that then $f_* = g_* : H_*(X; G) \to H_*(Y; G)$ for any coefficient group G. Show by example that this is false.