Workshop 2:
Axioms of homology

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Let $\mathbf{Top_2}$ denote the category whose objects are pairs of topological spaces (X, A) with $A \subset X$, and whose arrows (morphisms) $f : (X, A) \to (Y, B)$ are given by continuous maps $f : X \to Y$ such that $f(A) \subset B$. We write X instead of $(X, \emptyset)$.
An *homology theory* is given by functors $\{H_n : \mathbf{Top_2} \to \mathbf{Ab}\}_{n\in\mathbb{N}}$ and natural transformations $\{\partial_* \equiv \partial_*^n : H_n \Rightarrow H_{n-1} \circ \pi_2\}_{n\in\mathbb{N}}$, where $\pi_2(X, A) = (A, \emptyset)$. Each functor H_n maps a pair (X, A) to an abelian group $H_n(X, A)$ and $f : (X, A) \to (Y, B)$ to an homomorphism of groups $H_n(f) \equiv f_* : H_n(X, A) \to H_n(Y, B)$. The data are subject to the following axioms:

1. *Homotopy axiom:* If $f \simeq g : (X, A) \to (Y, B)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$ for every n.
2. *Exactness axiom:* For the inclusions $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$, the sequence
$$\cdots \xrightarrow{\partial_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_*} \cdots \tag{1}$$
is exact.
3. *Excision axiom:* Given a pair (X, A) and an open set $U \subset X$ such that $\bar{U} \subset \operatorname{int}(A)$, the inclusion $k : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism
$$k_* : H_n(X \setminus U, A \setminus U) \to H_n(X, A) \tag{2}$$
for every n.
4. *Dimension axiom:* For a one-point space P, $H_i(P) = 0$ for all $i \neq 0$.

Solve the following problems using this axiomatic characterization of homology.

Exercise 1 (Homotopy invariance)**.** Prove that if (X, A) and (Y, B) are homotopy equivalent,[1] then $H_n(X, A) \cong H_n(Y, B)$ for all n.

Exercise 2 (Finite additivity)**.** Let $X + Y$ denote the topological sum, and let $\iota_X : X \hookrightarrow X + Y$ and $\iota_Y : Y \hookrightarrow X + Y$ denote the canonical inclusions.[2] Prove that, for each n, the induced map
$$\iota_{X*} \oplus \iota_{Y*} : H_n(X) \oplus H_n(Y) \to H_n(X + Y) \tag{3}$$
is an isomorphism.
(Hint: Consider the exact sequence associated with the inclusion $X \hookrightarrow X + Y$.)

Exercise 3. Prove that for every space X and every integer n, $H_n(X, X) = 0$.

[1] This is, there exists a map $f : X \to Y$ such that $f(A) \subset B$ with homotopy inverse $g : Y \to X$ such that $g(B) \subset A$

[2] The underlying set of $X + Y$ is the disjoint union of the sets X and Y. A subset U of $X + Y$ is open if and only if its inverse images $\iota_X^{-1}(U)$ and $\iota_Y^{-1}(U)$ are open.

Exercise 4 (Reduced homology)**.** Given $X \neq \emptyset$, consider the unique map $\epsilon : X \to P$. Given any $\iota : P \to X$, we have $\epsilon \circ \iota = \mathrm{id}$, hence ϵ_* is onto. (Why?) We define the reduced homology $\tilde{H}$ via the equations $\tilde{H}_0(X) = \ker \epsilon_*$, $\tilde{H}_i(X) = H_i(X)$ when $i \neq 0$, and $\tilde{H}_i(X, A) = H_i(A)$ for any i when $A \neq \emptyset$.
The maps $i : A \hookrightarrow X$, $j : X \hookrightarrow (X, A)$ and $\epsilon : (X, A) \to (P, P)$ induce a diagram (of solid lines)

$$\begin{array}{ccccccccc}
 & & H_0(A) & \dashrightarrow & H_0(X) & & & & \\
 & \overset{\bar{\partial}_*}{\nearrow} & \downarrow & & \downarrow & \searrow & & & \\
H_1(X,A) & \xrightarrow{\partial_*} & H_0(A) & \xrightarrow{i_*} & H_0(X) & \xrightarrow{i_*} & H_0(X,A) & \xrightarrow{\partial_*} & H_{-1}(A) \\
\downarrow{\scriptstyle \epsilon_*} & & \downarrow{\scriptstyle \epsilon_*} & & \downarrow{\scriptstyle \epsilon_*} & & \downarrow{\scriptstyle \epsilon_*} & & \\
H_1(P,P) = 0 & \xrightarrow{\partial_*} & H_0(P) & \xrightarrow{\mathrm{id}_*} & H_0(P) & \longrightarrow & H_0(P,P) = 0 & &
\end{array}$$

1. Show that $i_*(\tilde{H}_0(A)) \subset \tilde{H}_0(X)$.
2. Show that $\mathrm{im}\, \partial \subset \tilde{H}_0(A)$, and hence that there is a well defined map $\bar{\partial} : H_1(X, A) \to \tilde{H}_0(A)$.
3. Show that there the sequence

$$H_1(X, A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X, A) \tag{4}$$

   of induced maps (dotted arrows in the diagram) is exact and thus there is a long exact sequence of reduced homology groups for any pair (X, A).

Exercise 5 (Homology of spheres)**.** We denote $H_0(P)$ by G.

1. Show that $\tilde{H}_0(S^0) = G$ and that $\tilde{H}_i(S^0) = 0$ for $i \neq 0$.
2. Show that, for any $n \geq 1$ and any integer i, $H_i(D^n, S^{n-1}) \cong H_{i-1}(S^{n-1})$.
   (Hint: a long exact sequence of reduced homology groups.)
3. Prove that for any $n \geq 1$ and any integer i, $H_i(S^n) \cong H_i(S^n, D^n_+)$, where D^n_+ denotes the "northern" hemisphere of the standard sphere S^n.
4. Prove that $H_i(S^n, D^n_+) \cong H_i(D^n, S^{n-1})$.
   (Hint: Excision.)
5. What is the reduced homology of the sphere S^n?