Assignment 5

August 5, 2024

1 Coding map

The following two exercises refer to the map Q_c that we studied in Chapter 6 of the notes. We follow the notations there.

1. Suppose $c < -(5 + 2\sqrt{5})/4$. Prove by induction in n that $I_{s_0,\dots,s_n}$ is a closed interval contained in $I_{s_0,\dots,s_{n-1}}$ and that

$$|I_{s_0,\dots,s_n}| < \eta^{-n} K, \tag{1}$$

where η is a lower-bound for $|Q_c'(x)|$ on $I_0 \cup I_1$, and $K = \max(|I_0|, |I_1|)$.

2. Prove that h^{-1} is continuous.

2 Two-dimensional systems

Adapted from Alligood et al.

Exercise 1. Consider the Hénon map

$$f_{a,b}(x, y) = (a - x^2 + by, x), \tag{2}$$

where a and b are real constants.

Hénon map Under which conditions $f_{a,b}$ has fixed points?

Hénon mbp Set $b = 0.4$.

(a) Prove that for $-0.09 < a < 0.27$, the Hénon map $f_{a,b}$ has one sink fixed point and one saddle fixed point.

(b) Find the highest magnitude eigenvalue of the Jacobian matrix at the first fixed point when $a = 0.27$. Explain the loss of stability of the sink.

(c) Prove that for $0.27 < a < 0.85$, $f_{a,b}$ has a period-two sink.

(d) Find the largest magnitude eigenvalue of $Df^2_{a,b}$, the Jacobian of $f^2_{a,b}$, at the period-two orbit, when $a = 0.85$.

Hénon mcp Prove that the Hénon map $f_{a,b}$ has a period-two orbit if and only if $4b > 3(1 - b)^2$.

Exercise 2 (Stable and unstable manifold)**.** Consider the fixed point 0 of the map $f(x, y) = (x/2, 2y - 7x^2)$.

1. Find the inverse map f^{-1}.

2. Show that the set $S = \{(x, 4x^2) : x \in \mathbb{R}\}$ is invariant under f, that is: if $v \in S$, then $f(v)$ and $f^{-1}(v)$ are in S too.

3. Show that each point in S converges to 0 under f.

4. Show that no point outside of S converges to 0 under f.

Try solving (a subset of) exercises 2.1 - 2.7 and Challenge 2 in Alligood et al.