California Institute of Technology
Ma 4 - Introduction to Mathematical Chaos

Spring 2024
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Some suggested problems for assignment 3

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Besides these exercises, you could work on Challenge 1 in Alligood et al.
Recall that $\mathbb{N}^* = \{1, 2, 3, ...\}$.

1 Transition graphs

Consider a continuous function $F : I \to \mathbb{R}$ defined on a compact interval I. A **partition** of I is a finite collection of compact intervals $\mathcal{I} = \{I_i\}$ such that $\bigcup_i I_i = I$ and $I_i \cap I_j$ is empty or contains a single point whenever $i \neq j$.

A **transition graph** is a graph G whose vertices are the intervals I_i and that has an arrow $I_i \to I_j$ iff $I_j \subset F(I_i)$, that is if F can take a point from I_i to I_j.

Exercise 1. Prove that if $S_1 \to S_k \to S_{k-1} \to \cdots \to S_2 \to S_1$ is an allowable path in G, then F^k has a fixed point x_0 in S_1 such that $F^i(x_0) \in S_{k-(i-1)}$ for all $i \in \{1, ..., k\}$.

(Remark about notation: The symbols S_j stand for vertices of the graph G, hence each S_j is one of the intervals in the partition $\mathcal{I}$. The indexes of the Ss simply correspond to an enumeration of the vertices in the path. The arrows correspond to any edge, so we are not excluding that $S_j = S_{j-1}$ for some j.)

Exercise 2. Each graph in Figure 1 represents a piece-wise linear function (along with $y = x$ and a grid). In each case, there is a period 4 cycle given by $\{0, 1, 2, 3\}$. One of these functions has cycles of all other periods, and one has only periods 1, 2, and 4. Identify which function has each of these properties. Justify your answer.

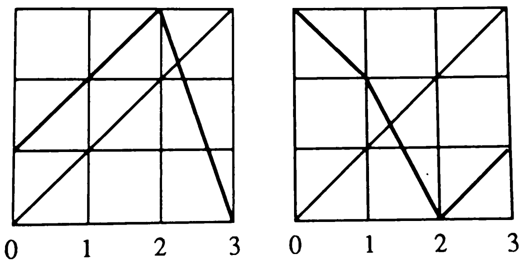

Figure 1: Two functions with a 4-cycle.

2 Cantor set

Recall that for any $(c_1, ..., c_n) \in \{0, 1, 2\}^n$,

$$C_{c_1,...,c_n} = \{\, x = 0.{}_3 d_1 d_2 d_3 \cdots \in [0, 1] \,:\, d_i = c_i \text{ for each } i = 1, ..., n \,\}. \tag{1}$$

Exercise 3. Prove rigorously that

$$\bigcap_{n\in\mathbb{N}^*}\left(\bigcup_{(c_1,\dots,c_n)\in\{0,2\}^n} C_{c_1,\dots,c_n}\right) = \bigcup_{(c_1,c_2,\dots)\in\{0,2\}^{\mathbb{N}^*}}\left(\bigcap_{n\in\mathbb{N}} C_{c_1,\dots,c_n}\right). \tag{2}$$

(Hint: To prove the equality of sets $A = B$, you should show that $A \subset B$ and $B \subset A$.)

3 Dynamics on the Cantor set

Exercise 4. Consider the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = 3x$ if $x \leq \frac{1}{2}$ and $T(x) = 3 - 3x$ if $x > \frac{1}{2}$.

1. Sketch the graph of T and show by graphical analysis that if $x > 1$ or $x < 0$, then $T^n(x) \to -\infty$ as $n \to +\infty$.

2. Let $\Gamma = \{\, x \in [0,1] \, : \, T^n(x) \in [0,1] \text{ for all } n \in \mathbb{N}^* \,\}$. Prove that $\Gamma = K$, the Cantor middle-thirds set. (Hint: Consider first the set $I_1 = \{\, x \in [0,1] \, : \, T(x) \in [0,1] \,\}$. What does T^2 look like on this set?)

3. Suppose $x \in \Gamma$ has ternary expansion $0._3 a_1 a_2 a_3 \dots$. What is the ternary expansion of $T(x)$? Be careful: there are two very different cases! (Remark that the coefficients of the ternary expansion must be 0, 1 or 2.)