Workshop 1: Fixed and periodic points, bifurcations

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Exercise 1. Consider the *doubling function* $D : [0,1) \to [0,1)$ defined by

$$D(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x - 1 & \text{if } 1/2 \le x < 1 \end{cases}. \tag{1}$$

In other words, $D(x) = 2x \mod 1$.

1. Find explicit formulas for D^2 and D^3.
2. Plot D, D^2 and D^3. How should the graph of D^n look like?
3. Find the fixed points of D^2 and D^3. How many fixed points does D^n have?

Exercise 2. Let f be a continuous map defined on the real line $\mathbb{R}$ such that f' exists everywhere and is continuous too. Suppose that f has a fixed point x_0 such that $|f'(x_0)| > 1$. Show that there exists a neighborhood $N_a(x_0) := (x_0 - a, x_0 + a)$ with the following property: for all $x \in N_a(x_0) \setminus \{x_0\}$ there is an integer n such that $f^n(x) \notin N_a(x_0)$.

Exercise 3. Find a tangent bifurcation and a period-doubling bifurcation for $F_a(x) = a \sin(x)$.

Exercise 4. Consider the parametric family of functions $F_\lambda(x) = \lambda x(1-x)$, where the parameter λ is a real number. For which values of λ does F_λ have a stable fixed point at $x = 0$? For which values of λ does F_λ has a nonzero stable fixed point? Describe the bifurcation that happens at $\lambda = 1$. Sketch the bifurcation diagram near $\lambda = 1$.

Exercise 5. Each of the following functions undergoes a bifurcation at the given parameter value. In each case, use algebraic or graphical methods to identify the bifurcation as either tangent or period-doubling, or neither of these. Describe the orbits of any seed x_0 for parameter values below, at, and above the bifurcation value.

(a) $F_\lambda(x) = x + x^2 + \lambda$ at $\lambda = 0$.

(b) $F_\lambda(x) = x + x^2 + \lambda$ at $\lambda = -1$.

(c) $G_\mu(x) = \mu x + x^3$ at $\mu = -1$.

(d) $G_\mu(x) = \mu x + x^3$ at $\mu = 1$.

Exercise 6. Consider the parametric family of functions $Q_c(x) = x^2 + c$, where the parameter c is a real number. Find formulas for the period-2 points q_+ and q_- that arise with the bifurcation at $c = -3/4$. Prove that this 2-cycle is attracting for $-5/4 < c < -3/4$, neutral for $c = -5/4$, and repelling for $c < -5/4$.