Worksheet

August 5, 2024

1 Subshifts

Consider the set

$$\Sigma' = \{(s_0, s_1, s_2, ...) \in \Sigma \; : \; \text{for every } j \in \mathbb{N}, \text{ if } s_j = 0 \text{ then } s_{j+1} = 1\}. \tag{1}$$

Remark that if s belongs to Σ', then $\sigma(s)$ also belongs to Σ'.

1. Prove that periodic points (in Σ') are dense in Σ'.
   (Hint: We proved this for Σ. What could fail here? How do you fix that?)

2. Prove that if $s \notin \Sigma'$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap \Sigma' = \emptyset$

3. Prove that there is a dense orbit in Σ'.

4. Prove that $\sigma : \Sigma' \to \Sigma'$ is chaotic.

2 The tent map

Consider the map $V(x) = 2|x| - 2$.

1. Sketch the graph of V, together with the line $y = x$. Use graphical analysis to show $V^n(x) \to \infty$ when $x \notin [-2, 2]$.

2. Sketch the graph of V^2 and V^3 on $[-2, 2]$. How is the graph of V^n?

3. Prove that the system $V : [-2, 2] \to [-2, 2]$ is chaotic.

We introduce now the map $C : [-2, 2] \to \mathbb{R}$, $x \mapsto -2 \cos\left(\frac{\pi x}{2}\right)$.

4. Sketch the graph of C on $[-2, 2]$. What is the range of C? Is C injective?

5. Prove that $C \circ V = Q_{-2} \circ C$.

6. Using 3 and 5, prove that Q_{-2} is chaotic.

3 Chaos is preserved under conjugation

Let (X, d_X) and (Y, d_Y) be metric spaces, and $F : X \to X$ and $G : Y \to Y$ dynamical systems. A *conjugation* between F and G is an homeomorphism[1] $h : X \to Y$ such that $G \circ h = h \circ F$. Prove that the property of being chaotic is preserved under conjugation.

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[1] An homeomorphism is a continuous and invertible map whose inverse is also continuous.