Workshop 3: Two-dimensional systems

August 5, 2024

Adapted from Alligood et al., Chapter 2.

Exercise 1 (Hénon map)**.** Consider the Hénon map

$$f_{a,b}(x, y) = (a - x^2 + by, x), \tag{1}$$

where a and b are real constants.

1. Under which conditions $f_{a,b}$ has fixed points?
2. Set $b = 0.4$.
   (a) Prove that for $-0.09 < a < 0.27$, the Hénon map $f_{a,b}$ has one sink fixed point and one saddle fixed point.
   (b) Find the highest magnitude eigenvalue of the Jacobian matrix at the first fixed point when $a = 0.27$. Explain the loss of stability of the sink.
   (c) Prove that for $0.27 < a < 0.85$, $f_{a,b}$ has a period-two sink.
   (d) Find the largest magnitude eigenvalue of $Df^2_{a,b}$, the Jacobian of $f^2_{a,b}$, at the period-two orbit, when $a = 0.85$.
3. Prove that the Hénon map $f_{a,b}$ has a period-two orbit if and only if $4b > 3(1 - b)^2$.

Exercise 2 (Stable and unstable manifold)**.** Consider the fixed point 0 of the map $f(x, y) = (x/2, 2y - 7x^2)$.

1. Find the inverse map f^{-1}.
2. Show that the set $S = \{(x, 4x^2) : x \in \mathbb{R}\}$ is invariant under f, that is: if $v \in S$, then $f(v)$ and $f^{-1}(v)$ are in S too.
3. Show that each point in S converges to 0 under f.
4. Show that no point outside of S converges to 0 under f.