Workshop 4: Dimension and entropy

August 5, 2024

Exercise 1 (Markov measure; cf. Pesin, Lectures 14-15)**.** Given $k \in \mathbb{N}$, consider the space

$$\Sigma_k^+ = \{\, w = (w_1, w_2, ...) \,:\, w_i \in \{1, ..., k\} \,\} \tag{1}$$

Let $P = (p_{i,j})_{i,j=1}^k$ be a matrix with nonnegative entries and such that

$$\forall i = 1, ..., k, \quad \sum_{j=1}^{k} p_{i,j} = 1, \tag{2}$$

and let $\pi = (\pi_1, ..., \pi_k)$ be a probability row vector that satisfies $\pi P = \pi$.

The matrix P encodes probabilities of transitions between states, and π is a stationary probability distribution (under those transitions).

1. Set
$$\ell(C_{w_1,...,w_n}) = \pi_{w_1} P_{w_1,w_2} P_{w_2,w_3} \cdots P_{w_{n-1},w_n}. \tag{3}$$
Use (2) to prove that ℓ is additive on cylinders. (When is a union of cylinders itself a cylinder?)

2. Show that the *Markov measure* μ induced by ℓ via the Carathéodory construction, namely by the outer measure
$$\mu^*(A) = \inf\{ \sum_i \ell(C_i) \,:\, \bigcup_i C_i \supset A,\, C_i \text{ cylinder} \}, \tag{4}$$
is shift-invariant.

3. Define a *transition matrix* $A = (a_{i,j})$ by the formula
$$a_{i,j} = \begin{cases} 1 & \text{if } p_{i,j} > 0 \\ 0 & \text{if } p_{i,j} = 0 \end{cases} \tag{5}$$
A is primitive if there exists n such that all entries of A^n are positive. A sequence $w \in \Sigma_k^+$ is *admissible* if $a_{w_j,w_{j+1}} =$ for all j. (The set of admissible sequences is Σ_A^+.)
Prove that if A is primitive as above, then for every $m \geq n$ and any $i, j \in \{1, ..., k\}$, there exists an admissible word w with $w_1 = i$ and $w_{m+1} = j$.

4. Prove that the support of μ,
$$\operatorname{supp} \mu = \{\, x \in X \,:\, \mu(E) > 0 \text{ for any open } E \text{ that contains } x \,\}, \tag{6}$$
is the set of admissible sequences Σ_A^+.

Exercise 2 (Entropy of Markov measures)**.** We keep the notation from the previous exercise.

Consider the following distance on Σ_k^+:

$$\forall s, t \in \Sigma_k^+, \quad d_a(s,t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{a^i} \tag{7}$$

for some $a > 2$.

1. Given $w = (w_1, w_2, ...)$, prove that the cylinder $C_{w_1,\ldots,w_n}$ is exactly the ball of radius $1/a^n$ at w.

2. Prove that

$$C_{w_1,\ldots,w_n} = \{\, v \in \Sigma_k^+ \,:\, d_a(\sigma^j(v), \sigma^j(w)) < 1/a \text{ for all } 0 \leq j < n \,\}. \tag{8}$$

3. For a dynamical system $f : X \to X$ on a metric space (X, d), a Bowen ball is given by

$$B_f(x, n, \delta) = \{\, y \in X \,:\, d(f^j(x), f^j(y)) < \delta \text{ for all } 0 \leq j < n \,\} \tag{9}$$

and if ν is a measure on X, its local entropy is

$$h_{\nu,f}(x) = \lim_{\delta \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu(B_f(x, n, \delta)). \tag{10}$$

Prove that

$$h_{\mu,\sigma}(w) = -\sum_{i=1}^{k} \pi_i \sum_{j=1}^{k} p_{i,j} \log p_{i,j}. \tag{11}$$

Exercise 3 (Pesin, Ex. 4.6)**.** Show that if f is a piecewise linear one-dimensional Markov map (not necessarily fully branched) with contact expansion $|f'(x)| = 1/\lambda$, then the pointwise dimension and the local entropy exist at precisely the same points, and they are related by

$$d_\mu(x) = \frac{h_{\mu,f}(x)}{-\log \lambda}. \tag{12}$$